Identifiability in multiple time series

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SUMMARY.

This paper deals with identifiability of matrixcoëfficiënts in multivariate stochastic difference-equation-models, where identifiability should be interpreted as in [5], that is, we are only concerned with identifiability with respect to the class $\mathcal{P}^{(n)}$, the class of possible distributions of a sample taken from the observable process at n consecutive time points. (For a discussion on this: see [2] and [5]).

CONTENTS.

1. Introduction.

In this paper we study m-variate stochastic processes with complex-valued components and finite second moments, which are assumed to be the unique (a.s.) weakly stationairy solution of a difference-equation of the form:

$$(1.1) \qquad \sum_{k=0}^{p} A_k \underline{x}_{t-k} = \sum_{j=0}^{q} B_j \underline{\varepsilon}_{t-j} \qquad t = 0,\pm 1,\pm 2,\ldots$$

where $\{\underline{\varepsilon}_t\}$ is (unobservable) m-variate white noise, with unknown covariance-matrix Σ_ε. Furthermore, the m×m matrices A_k, $k = 1,\ldots,p$ and B_j, $j = 1,\ldots,q$ are supposed to be unknown. As in [5] we introduce a parameter ξ, characterizing the distribution of the process $\{\underline{\varepsilon}_t\}$, and we put:

$$\Theta = (A_1,\ldots,A_p \quad ; \quad B_1,\ldots,B_q,\xi)$$

In section 3 identifiability will be proved for

$$\Psi(\Theta) = (A_1,\ldots,A_p \quad , \quad B_1,\ldots,B_q, \Sigma_\varepsilon(\xi))$$

for the special case p = 0 (Moving Average).
In section 5 the general case will be treated.

2. Notations and preliminary results.

Let $\{\underline{x}_t\}$ be a m-variate weakly stationary process. Then its matrix-valued spectral measure (see: [3]) will be denoted by F_x and its spectral representation by:

$$\int e^{it\lambda} \underline{z}_x \{d\lambda\}$$

where $\underline{z}_x(\lambda)$ is the associated m-variate process with orthogonal increments with:

$$E\{\underline{x}_x\{d\lambda\}\ \underline{z}_x^{*}\{d\lambda\}\} = F_x\{d\lambda\}.$$

(The $*$ indicates the complex conjugate transpose). Furthermore we introduce the real-valued measure:

$$\Phi_x\{d\lambda\} = \operatorname{tr} F_x\{d\lambda\}$$

It is easily seen that $\Phi_x\{D\} = 0$ i.f.f. (if and only if) $F_x\{D\} = 0$. Thus we can write:

$$F_x\{d\lambda\} = \tilde{f}_x(\lambda)\ \Phi_x\{d\lambda\}.$$

When the measure Φ_x is absolutely continuous with respect to Lebesgue measure, (with density $\varphi_x(\lambda)$), F_x is said to be so and we can write:

$$F_x\{\lambda\} = f_x(\lambda)d\lambda$$

where $f_x(\lambda) = \tilde{f}_x(\lambda)\ \varphi_x(\lambda)$ is called the spectral density matrix.

When we are dealing with processes, satisfying the difference equation (1.1), it is usefull to define the matrix-generating functions:

$$A(z) = \sum_{k=0}^{p} A_k\ z^k,$$

$$B(z) = \sum_{k=0}^{q} B_k\ z^k.$$

To achieve maximum analogy between the univariate and the multivariate case, we introduce the concept of a singularity-point:

Definition 2.1.

Let $Q(z)$ be a square matrix-valued function of the complex variable z. Then the point z_0 is zalled a singularity point of $Q(z)$ if $\det Q(z_0) = 0$.

Definition 2.2.

A matrixvalued function of the form $\sum_{k=0}^{p} A_k z^k$ is called a matrixpolynomial; if $A_p \neq 0$ p is the degree of the matrixpolynomial.

It will be clear now, that singularitypoints of matrixpolynomials will take over the role of the zeros of polynomials in the univariate case.

Definition 2.3.

The matrixfunction $Q(z)$ is called analytic (or holomorphic), if all its elements are analytic functions of z.

Since singularity points of an analytic matrix $Q(z)$ are in fact zeros of the analytic function $\det Q(z)$, they are either isolated points, and hence the set of singularity points has lebesgue-measure 0, or $\det Q(z)$ is identically zero.

We can now state:

Theorem 2.1.

The homogeneous difference-equation

$$(2.1.) \quad \sum_{k=0}^{p} A_k \underline{x}_{t-k} = 0 \qquad t = 0, \pm 1, \pm 2, \ldots$$

has a non-trivial weakly stationary solution i.f.f. $A(e^{-i\lambda})$ has as a function of λ at least one singularitypoint. Furthermore, if S is the set of singularity points and S' its complement, we have for all solutions $\{\underline{x}_t\}$:

$$F_x\{S'\} = 0$$

Proof. Suppose:

$$\underline{x}_t = \int e^{it\lambda} \; \underline{z}_x\{d\lambda\}$$

is a weakly stationary solution of (2.1), thus:

$$\int A(e^{-i\lambda}) \; e^{it\lambda} \; \underline{z}_x\{d\lambda\} \overset{a.s}{=\!=} 0 \Leftrightarrow$$

$$\int A(e^{-i\lambda}) \; F_x\{d\lambda\} \; A^*(e^{-i\lambda}) = 0 \Leftrightarrow$$

$$\int A(e^{-i\lambda}) \; \tilde{f}_x(\lambda) \; A^*(e^{-i\lambda}) \; \Phi_x\{d\lambda\} = 0 \Leftrightarrow$$

$$A(e^{-i\lambda}) \; \tilde{f}_x(\lambda) \; A^*(e^{-i\lambda}) \overset{a.e}{=\!=} 0, \text{ w.r.t. } \Phi_x \text{ (or } F_x\text{)}.$$

Hence, for $\lambda \in S'$ we must have:

$$\tilde{f}_x(\lambda) \overset{a.e}{=\!=} 0 \text{ w.r.t. } \Phi_x \Rightarrow$$

$$F_x\{S'\} = \int_{S'} F_x\{d\lambda\} = \int_{S'} \tilde{f}_x(\lambda) \; \Phi_x\{d\lambda\} = 0$$

<u>Remark:</u> In contrast to the univariate case, the m-variate homogeneous difference equation may have a non trivial solution with absolutely continuous spectrum, due to the fact that det $A(e^{-i\lambda})$ can be identically zero.

3. The basic lemma.

Lemma 3.1.

Let $Q(z)$ be a m×m matrixpolynomial of degree $p > 0$, which is nonsingular for at least one point z_1, and let Σ be an arbitrary hermitian positive definite m×m matrix.
Then there exists a matrixpolynomial $Q_0(z)$ with:

a) $Q_0(z)$ nonsingular for $|z| < 1$.

b) $Q(e^{-i\lambda})\ \Sigma\ Q^*(e^{-i\lambda}) = Q_0(e^{-i\lambda})\ \Sigma\ Q_0^*(e^{-i\lambda})\ \forall_{\lambda \in [-\pi,\pi]}$

Furthermore $Q_0(z)$ is of degree p and is uniquely determined except for a constant m×m matrix H, with $H\ \Sigma\ H^* = \Sigma$.

Proof. Since Σ can be decomposed into TT^*, and the matrix T can be absorbed into $Q(z)$, it is no restriction to take $\Sigma = I_m$, the m×m unit-matrix.

▶ Existance.

Let z_0 be a singularitypoint of $Q(z)$ with $|z_0| < 1$, and let $c_1,\dots,c_k$ be an orthonormal basis for $\ker Q(z_0)$. The matrix with columns $c_1,\dots,c_k$ is denoted by C.
Let $d_1,\dots,d_{m-k}$ be an orthonormal basis for $[\ker Q(z_0)]^{\perp}$, the orthogonal complement of $\ker Q(z_0)$ and denote the matrix with columns $d_1,\dots,d_{m-k}$ by D.
Put:

$$U = [\,C;D]$$

then we have:

$$U^*U = UU^* = I_m.$$

Since:

$$Q(z_0)C = 0$$

there exists an integer $a_0 > 0$, such that:

$$Q(z)\ C = (z-z_0)^{q_0}\ Q_1(z) \quad \forall z,$$

where $Q_1(z)$ is a m×m matrix-polynomial of degree $p-a_0$ and $\operatorname{rank}(Q_1(z_0)) = k$. Thus:

$$Q(z)U = [\,(z-z_0)^{a_0}\ Q_1(z);\ Q(z)D]\ .$$

Post-multiplying by U^* and using (2.2) yields:

$$Q(z) = [\,(z-z_0)^{a_0}\ Q_1(z);\ Q(z)D]\ U^*$$

Put:

$$\tilde{Q}(z) = [\,(z\bar{z}_0-1)^{a_0}\ Q_1(z);\ Q(z)D\,]\,U^*$$

Then we have:

$$\det \tilde{Q}(z) = (z\bar{z}_0-1)^{a_0 k}\ \det [\,Q_1(z);\ Q(z)D]\ \det U^*$$

$$= \left(\frac{z\bar{z}_0-1}{z-z_0}\right)^{a_0 k}\ \det Q(z).$$

Thus, ignoring multilicities, $\tilde{Q}(z)$ has at least one singularity point less than $Q(z)$ in $|z| < 1$. Furthermore:

$$\tilde{Q}(e^{-i\lambda})\tilde{Q}^*(e^{-i\lambda}) = |e^{-i\lambda}\bar{z}_0-1|^{2a_0}\ Q_1(e^{-i\lambda})Q_1^*(e^{-i\lambda}) + Q(e^{-i\lambda})DD^*Q^*(e^{-i\lambda})$$

$$= |e^{-i\lambda}-z_0|^{2a_0}\ Q_1(e^{-i\lambda})Q_1^*(e^{-i\lambda})+Q(e^{-i\lambda})DD^*Q^*(e^{-i\lambda})$$

$$= Q(e^{-i\lambda})\ Q^*(e^{-i\lambda}),$$

so that $\tilde{Q}(z)$ satisfies condition b). Since this procedure can be repeated for every singularity point, we can find in finitely many steps a matrix $Q_0(z)$ of degree p satisfying a) and b).

▶ Uniqueness.

Suppose we have two matrixpolynomials Q_0 and P_0, both of degree p, satisfying a) and b). If z_0 is a singularitypoint of Q_0 and P_0 with $|z_0| = 1$, then b) implies:

$$\ker Q_0^*(z_0) = \ker P_0^*(z_0).$$

Thus we can write as before:

$$Q_0^*(z) = [((z-z_0)^{a_0} Q_1(z))^*, \; Q_0^*(z)D\,]U^*$$

$$P_0^*(z) = [\,((z-z_0)^{b_0} P_1(z))^*, \; P_0^*(z)D\,]U^*$$

where $\det [\, Q_1^*(z_0), \; Q_0^*(z_0)D] \neq 0$ and

$$\det [\, P_1^*(z_0), \; P_0^*(z_0)D] \neq 0$$

Since b) implies:

$$|\det Q_0^*(e^{-i\lambda})| = |\det P_0^*(e^{-i\lambda})| \quad , \forall\lambda$$

it is easily seen that we must have $a_0 = b_0$.
Therefore it suffices to prove the uniqueness of:

$$W(z) = \begin{bmatrix} Q_1(z) \\ \\ D^* \; Q_0(z) \end{bmatrix}$$

Put:

$$T(z) = \begin{bmatrix} P_1(z) \\ \\ D^* \; P_0(z) \end{bmatrix}$$

Then we have:

$$Q(z) = \mathcal{V}(z) \; W(z), \; P_0(z) = \mathcal{V}(z) \; T(z)$$

where:

$$\mathcal{V}(z) = \left[\begin{array}{c|c} (z-z_0)^{a_0} I_k & 0 \\ \hline 0 & I_{m-k} \end{array}\right]$$

Thus for all z we have:

$$W(z) = \lim_{\eta \to z} \mathcal{V}^{-1}_{(\eta)}\, Q_0(z),\ T(z) = \lim_{\xi \to z} \mathcal{V}^{-1}_{(\xi)}\, P_0(z).$$

Hence:

$$W(e^{-i\lambda})W^{*}(e^{-i\lambda}) = \lim_{\eta \to e^{-i\lambda}} \mathcal{V}^{-1}_{(\eta)}\, Q_0(e^{-i\lambda})Q_0^{*}(e^{-i\lambda})\, \mathcal{V}^{-1^{*}}_{(\eta)} =$$

$$= \lim_{\xi \to e^{-i\lambda}} \mathcal{V}^{-1}_{(\xi)} P_0(e^{-i\lambda})P_0^{*}(e^{-i\lambda}) \mathcal{V}^{-1^{*}}_{(\xi)}$$

$$= T(e^{-i\lambda})T^{*}(e^{-i\lambda}).$$

Thus $W(z)$ and $T(z)$ also satisfy b), but have at least one singularity point less than $P_0(z)$ and $Q_0(z)$ on $|z| = 1$.
In this way we can remove all singularitypoints from the unitcircle, and therefore it is no restriction to assume $Q_0(z)$ and $P_0(z)$ non-singular on $|z| = 1$.
Consider:

$$V(z) = P_0^{-1}(z)\ Q_0(z)$$

Using b) we have:

$$V(e^{-i\lambda})V^{*}(e^{-i\lambda}) = P_0^{-1}(e^{-i\lambda})Q_0(e^{-i\lambda})Q_0^{*}(e^{-i\lambda})P_0^{*-1}(e^{-i\lambda}) =$$

$$= P_0^{-1}(e^{-i\lambda})P_0(e^{-i\lambda})P_0^{*}(e^{-i\lambda})P_0^{*-1}(e^{-i\lambda}) = I_m$$

or equivalently: $V(z)$ is unitary on $|z| = 1$. Furthermore $V(z)$ is analytic inside the unitcircle and has no singularitypoints on $|z| \leq 1$.
Therefore both $V(z)$ and $V^{-1}(z)$ can be expanded into a powerseries:

$$V(z) = \sum_{k=0}^{\infty} M_k z^k \quad , \qquad V^{-1}(z) = \sum_{k=0}^{\infty} N_k z^k, \ |z| < 1.$$

Since the elements of $V(z)$ and $V^{-1}(z)$ are rational functions with no poles on $|z| \leq 1$, both series converge on the unitcircle.
Hence:

$$V^{-1}(e^{-i\lambda}) = \sum_{k=0}^{\infty} N_k e^{-ik\lambda} = V^{*}(e^{-i\lambda}) = \sum_{k=0}^{\infty} M_k^{*} e^{ik\lambda} \Rightarrow$$

$$\Rightarrow \begin{cases} N_0 = M_0^{*} \\ N_k = M_k = 0, \ k \geq 1 \end{cases}$$

by a simple equation of coëfficiënts.
It follows that $V(z) = N_0$, or equivalently:

$$Q_0(z) = P_0(z)\, N_0, \text{ with } N_0 \text{ unitary.}$$

This proves the lemma.

As an immediate consequence of the lemma we have:

<u>Theorem 3.1.</u>

When in the m-variate Moving Average case the parameterspace θ is such that $B(z)$ has no singularitypoints in $|z| < 1$, and $B_0 = I_m$, then $\Psi(\Theta) = (B_1, \ldots, B_q, \Sigma_\varepsilon)$ is identifiable w.r.t. $\mathcal{P}^{(q+1)}$.

The proof is exactly that of the univariate case (see [5]) and will be omitted.

Remark: The distributions of the class $\mathcal{P}^{(q+1)}$ are in fact $m(q+1)$-dimensional distributions, and there are $m^2q + \frac{1}{2}m(m+1)$ unknown parameters. Thus for $m > 1$ one has identification with less observations than unknown parameters.

4. The AR-case.

Theorem 4.1.

When in the AR-case θ is such that $A_0 = I_m$, $A(z)$ has no singularitypoints on $|z| \leq 1$, and Σ_ε is non-singular, then:

$$\Psi(\theta) = (A_1 \ \dots \ A_p, \ \Sigma_\varepsilon)$$

is identifiable w.r.t. $\mathcal{P}^{(p+1)}$.

Proof.

As in the univariate case, we may write down the YULE-WALKER-equations:*)

$$(4.1) \qquad \Psi(\theta)\mathcal{R} = -R$$

where

$$R = (\Gamma_0, \dots, \Gamma_p) \ , \ \Gamma_s = E\{\underline{x}_t \ \underline{x}^*_{t-s}\}$$

and

$$\mathcal{R} = \begin{bmatrix} \Gamma_1^* & \Gamma_0 & \dots\dots & \Gamma_{p-1} \\ \Gamma_2^* & \Gamma_1^* & \dots\dots & \Gamma_{p-2} \\ \dots & \dots & \dots & \dots \\ \Gamma_r^* & \Gamma_{r-1}^* & \dots\dots & \Gamma_0 \\ -I_m & 0 & \dots\dots & 0 \end{bmatrix}$$

Thus we have identifiability when there is no non-trivial linear combination

$$\sum_{j=1}^{p} c_j' \ \underline{x}_{t-j} \quad , \quad c_j \in \mathbb{C}^m$$

with variance zero.

*) The condition that $A(z)$ has no singularity-points on $|z| \leq 1$ implies the existance of a non-sided MA-representation which in turn implies (4.1)

Suppose

$$(4.2) \qquad \sum_{j=1}^{p} c_j' \, \underline{x}_{t-j} \xrightarrow{a.s} 0$$

Since the process $\{\underline{x}_t\}$ has a spectral density given by:

$$f_x(\lambda) = A^{-1}(e^{-i\lambda}) \, \Sigma_\varepsilon \, A^{-1^*}(e^{-i\lambda})$$

we must have:

$$\sum_{j=1}^{p} \int c_j' \, A^{-1}(e^{-i\lambda}) \, \Sigma_\varepsilon \, A^{-1^*}(e^{-i\lambda}) \, c_j'^* \, d\lambda = 0$$

But then it follows easily that $c_j = 0$ $j = 1,\ldots,p$ since all terms are real and nonnegative.

Note that (4.2) does not imply a singular spectral measure (see § 2).

5. The ARMA-case.

The multivariate ARMA-case needs special attention. The generalization from the univariate case is far from obvious: matrixpolynomials don't factorize as simple as polynomials do, and there is no 1-1-correspondence between singularitypoints and factorizations; besides singularitypoints, also the corresponding nullspaces play an important role.
The following terminology will be usefull:

Definition 5.1.

The square matrices A and B are called comparable when they have the same nullspaces. They are called completely incomparable when the nullspaces have null-intersection.

HANNAN, who proved the identifiability with respect to the class $\mathcal{P}^{(\infty)}$ in [4], showed that a necessary condition for the matrixpolynomials $A(z) = \sum_{k=0}^{p} A_k z^k$ and $B(z) = \sum_{k=0}^{q} B_k z^k$ to have no common left factor*), is that A_p^* and B_p^* are completely incomparable, and he achieves identifiability by the condition that A(z) and B(z) have the unitmatrix as a greatest common left-divisor (g.c.l.d) (see mc. Duffee, [1] p 35).
The last condition is difficult to verify and a condition in terms of singularitypoints and nullspaces looks more appropriate.
The following lemma shows what kind of condition will be needed.

Lemma 5.1.

When the m×m-matrix polynomials A(z) and B(z) have a common singularitypoint z_0, and $A^*(z_0)$ and $B^*(z_0)$ are not completely imcomparable, then A(z) and B(z) have a common left factor.

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*) Obviously this is necessary for identifiability when only second-order properties are considered.

Proof:

Let C be the m×c matrix with columns an orthogonal basis for the intersection of the nullspaces of $A^*(z_0)$ and $B^*(z_0)$, and let D be the matrix with columns an orthonormal basis for the orthogonal complement.

In exactly the same way as in the proof of the basic lemma 3.1. we have:

$$U^*A(z) = \begin{bmatrix} (z-z_0)A_1(z) \\ D^*A(z) \end{bmatrix}$$

$$U^*B(z) = \begin{bmatrix} (z-z_0)B_1(z) \\ D^*\,B(z) \end{bmatrix}$$

where $U = [\,C\ ;\ D\,]$, $UU^* = U^*U = I_m$.

Hence:

$$A(z) = U \left[\begin{array}{c|c} (z-z_0)I_c & 0 \\ \hline 0 & I_{m-c} \end{array}\right] \begin{bmatrix} A_1(z) \\ D^*A(z) \end{bmatrix}$$

$$B(z) = U \left[\begin{array}{c|c} (z-z_0)I_c & 0 \\ \hline 0 & I_{m-c} \end{array}\right] \begin{bmatrix} B_1(z) \\ D^*\,B(z) \end{bmatrix}$$

which proves the lemma.

We now have:

Theorem 5.2.

When in the ARMA-case θ is such that:

a) $A_0 = B_0 = I_m$,

b) $A(z)$ has no singularitypoints on $|z| \leq 1$
   $B(z)$ has no singularitypoints on $|z| < 1$

c) $A^*(z)$ and $B^*(z)$ are completely imcomparable $\forall_z$
   A^*_p and B^*_q are completely incomparable.

then:

$$\Psi(\Theta) = (A_1,\dots,A_p,B_1,\dots,B_q,\Sigma_\varepsilon)$$

is identifiable w.r.t. $\mathcal{P}^{(p+q+1)}$.

<u>Proof.</u>

Put:

$$\Psi_1(\Theta) = (A_1,\dots,A_p),\ \Psi_2(\Theta) = (B_1,\dots,B_q,\Sigma_\varepsilon).$$

Postmultiplying both sides of the difference-equation (1.1) with $\underline{x}^*_{t-s}$, $s = q+1,\ q+2,\dots$ and taking expectations yields:

$$(5.1) \qquad \sum_{k=0}^{p} A_k\,\Gamma_{s-k} = 0 \quad , \quad s = q+1,q+2,\dots \qquad *)$$

where

$$\Gamma_s = E\{\underline{x}_t\ \underline{x}^*_{t-s}\}.$$

Since $A_0 = I_m$ this can for $s = q+1,\dots,q+p$ be written:

$$\Psi_1(\Theta)\mathcal{R}_q = -R'$$

where

$$R = \begin{bmatrix} \Gamma_{q+1} \\ \vdots \\ \Gamma_{q+p} \end{bmatrix} \quad \text{and} \quad \mathcal{R}_q = \begin{bmatrix} \Gamma_q & \Gamma_{q+1} & \cdots & \Gamma_{q+p-1} \\ \Gamma_{q-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Gamma_{q+1} \\ \Gamma_{q-p+1} & \Gamma_{q-1} & \cdots & \Gamma_q \end{bmatrix}$$

Suppose $\mathcal{R}_q$ is singular; then there exist vectors $c_0,\dots,c_{p-1}$, not all zero, such that:

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*) The condition that $A(z)$ has no singularitypoints on $|z| \leq 1$ implies that there exists a one-sided moving average representation, which in turn implies (5.1).

(5.2) $$\sum_{s=0}^{p-1} \Gamma_{j+s}\, c_s = 0 \qquad j = q-p+1,\ q-p+2,\dots,q$$

From (5.1) we have:

$$\Gamma_s = -\sum_{k=1}^{p} A_k\, \Gamma_{s-k} \qquad s = q+1, q+2, \dots$$

or equivalently:

$$\Gamma_{q+1+s} = -\sum_{k=1}^{p} A_k\, \Gamma_{q+1+s-k} \qquad s = 0,1,\dots,p-1$$

Postmultiplying by c_s and summing over s yields:

$$\sum_{s=0}^{p-1} \Gamma_{q+1-s}\, c_s = -\sum_{k=1}^{p} A_k \sum_{s=0}^{p-1} \Gamma_{q+1+s-k}\, c_s = 0$$

using (5.2). Continuing this way (5.2) is easily seen to be valid for $j \geq q-p+1$. Let

$$\underline{y}_t = \sum_{k=0}^{p-1} c_k^{*}\, \underline{x}_{t-k} \qquad t = 0, \pm 1, \pm 2, \dots$$

and

$$c(z) = \sum_{k=0}^{p-1} c_k^{*}\, z^k .$$

Then $\{\underline{y}_t\}$ is a weakly stationary process with spectral density matrix:

$$\frac{1}{2\pi}\, c(e^{-i\lambda})\, A^{-1}(e^{-i\lambda})\, B(e^{-i\lambda})\, \Sigma_{\varepsilon}\, B^{*}(x^{-i\lambda})\, A^{-1^{*}}(e^{-i\lambda})\, C^{*}(e^{-i\lambda})$$

(see: [3] § II.4).
(Here we use the fact that Σ_{ε} is nonsingular; otherwise $\underline{y}_t$ could be identically zero.)

$$E\{\underline{x}_{t-j}\, \underline{y}^{*}_{t-s}\} = \sum_{k=1}^{p-1} \Gamma_{s-j+k}\, c_k = 0 \quad , \quad s_j \geq q-p+1$$

Premultiplying with c_j^{*}, and summing over j yields:

$$E\{\underline{y}_t\, \underline{y}^{*}_{t+s}\} = 0 \quad , \quad |s| \geq q$$

This can equivalently be written as:

$$\int e^{is\lambda} c(e^{-i\lambda}) A^{-1}(e^{-i\lambda}) B(e^{-i\lambda}) \Sigma_{\varepsilon} B^{*}(e^{-i\lambda}) A^{-1^{*}}(e^{-i\lambda}) c^{*}(e^{-i\lambda}) d\lambda = 0, \quad |s| \geq q$$

But then it follows that $c(z)A^{-1}(z)B(z)$ must be a matrixpolynomial whose degree is at most $q-1$.

Let

$$K(z) = (\det A(z))\; A^{-1}(z)$$

then also $K(z)$ is a matrixpolynomial in z with:

$$\det K(z) = (\det A(z))^{m-1}$$

Denote the degree of $c(z)K(z)$ by s and the degree of $\det A(z)$ by a.
Since $c(z)A^{-1}(z)B(z)$ is a polynomial of degree at most $q-1$, say $q-r$, and $\det A(z)$ is a scalar polynomial, we must have:

$$s+q-a \geq q-r$$

thus:

$$a \leq s+r \quad , \qquad r \geq 1$$

1) Suppose: $a \leq s$.

Put:

$$s(z) = c(z)\; K(z) = \sum_{j=0}^{s} s_j\; z^j$$

Since $s(z)B(z)$ is a matrixpolynomial of degree at most $q-1+a$ we must have:

$$\sum_{k=0}^{s} s_k\; B_{l-k} = 0 \quad , \qquad l \geq q+a$$

In the same way for $s(z)\; A(z) = c(z) \det A(z)$ we must have:

$$\sum_{k=0}^{s} s_k\; A_{n-k} = 0, \qquad n \geq p+a$$

Choosing $l = q+s$ and $n = p+s$ yields:

$$s_s B_q = s_s A_p = 0$$

which contradicts the complete incomparability of A_p^* and B_q^*.

2) When $a \geq s+1$, it is easily seen that there exists at least one zero, say z_0 of det $A(z)$ with:

$$s(z_0) \neq 0$$

$$s(z)\ B(z) = (z-z_0)\ V(z),$$

for some polynomial $V(z)$.
Thus $B(z_0)$ is singular and $B^*(z_0)\ s^*(z_0) = 0$.

On the other hand we have:

$$s(z_0)\ A(z_0) = c(z_0) \det A(z_0) = 0,$$

thus also $A^*(z_0)s^*(z_0) = 0$, which contradicts the complete incomparability of $A(z)$ and $B(z)$ for all z, and completes the proof of the identifiability of $\Psi_1(\Theta)$.

We now fix $\Psi_1(\Theta)$ and put:

$$\underline{z}_t = \sum_{k=0}^{p} A_k\ \underline{x}_{t-k} = \sum_{j=0}^{q} B_j\ \underline{\varepsilon}_{t-j}$$

Thus $\{\underline{z}_t\}$ is a MA-process, with distribution independent of $\Psi_1(\Theta)$. If $\tilde{P}^{(n)}_{\Psi_2(\Theta)}$ denotes the joint distribution of $\underline{z}_1,\ldots,\underline{z}_n$, and

$$\tilde{\mathcal{P}}^{(n)} = \{\tilde{P}_{\Psi_2(\Theta)} \mid \Theta \in \boldsymbol{\theta}\},$$

Then we have the implication:

$$\tilde{P}^{(n)}_{\Psi_2(\Theta_1)} \neq \tilde{P}^{(n)}_{\Psi_2(\Theta_2)} \Rightarrow P^{(n+p)}_{\Theta_1} \neq P^{(n+p)}_{\Theta_2}$$

Thus since $\Psi_2(\Theta)$ is identifiable w.r.t. $\tilde{\mathcal{P}}^{(q+1)}$ (according to theorem 3.1), we also have identifiability w.r.t $\mathcal{P}^{(p+q+1)}$. This proves the theorem.

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